Bridge Dynamics in the 21st Century

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ABSTRACT: There is a wide spread misconception that the numerical methods employed in Computational Mechanics (The Finite Element Method, the Discrete Element Method and the Boundary Element Method) are recent developments of the 21st Century. The reality is that the backbone of the majority of these numerical techniques have been developed more than thirty years ago, being its earlier application limited to very small problems as a result of the lack of computational capacity of the then available personal computers. The huge advances in the computational power that have taken place in the last 15 years made feasible the use of these advanced analysis methods to model structures with ever-increasing levels of complexity. In the particular case of Bridge Dynamics, this has finally allowed bridge Engineers to approach the problems of moving vehicular loads on highway and railway bridges as a fully dynamic problem. This paper will focus on the available methods to perform such type of analysis with emphasis given to the relative advantages and shortcomings of each approach. This will be complemented by a real life case study, in which the author has been directly involved.

1 INTRODUCTION

1.1 *Theoretical background*

The main theoretical background of Structural Dynamics, also designated by Vibration Engineering, has been established more than a century ago. Since the second half of the 20th Century that numerical methods to solve the dynamical equations of motion (1) have been developed with notorious contributions from Newmark (Newmark 1969), Wilson (Wilson 1962), Bathe (Bathe 1982) amongst many others.

$$\mathbf{M}\cdot\ddot{\mathbf{u}}(t)+\mathbf{C}\cdot\dot{\mathbf{u}}(t)+\mathbf{K}\cdot\mathbf{u}(t)=\mathbf{F}(x,y,z,t) \tag{1}$$

In equation (1) the entities **M, C, K** and **F**(x,y,z,t) represent the global mass matrix, the global damping matrix, the global stiffness matrix and the global load vector, respectively.
The kinematical entities $\ddot{\mathbf{u}}$, $\dot{\mathbf{u}}$ (t) and **u**(t) are the acceleration, the velocity and the displacement vectors, respectively.

The main drive for the development of these techniques has been, nevertheless, attaining a better understanding of the structural behaviour of civil engineering structures when excited by ground borne vibrations such as the ones induced by earthquakes and the movement of underground trains.

Even in the case of bridge structures, the emphasis has been put mainly either on the dynamic behaviour of footbridges to pedestrian loading (in order to achieve acceptable levels of comfort) or in the structural response of decks and sub-structures to earthquakes to guarantee safety in terms of Ultimate Limit States.

When it comes to the dynamic analysis of Highway of Railway bridges, most of the available assessment and design codes still adopt a simplified approach when determining the additional deformations and internal forces due to moving loads, commonly referred to as dynamic amplification factors (DAF).

The exponential growth in computational power the world has witnessed in the last decade have finally made feasible for a structural engineer to conduct the numerical integration of the differential equations of motion that derive from modelling life size bridge structures, when undergoing moving vehicular loads that vary both in time, **F**(t), and also in space, **F**(x,y,z,t).

2 *METHODS TO DETERMINE THE DYNAMIC RESPONSE OF A BRIDGE TO MOVING VEHICULAR LOADS*

There are fundamentally two main ways to approach the problem of determining the response of a bridge to a moving vehicular loading:

- Frequency domain analysis
- Time domain analysis

2.1 *Frequency Domain Analysis*

In the frequency domain analysis, the dynamic action is decomposed into a number of periodic functions, using Fourier analysis.

The response of the structure is therefore obtained from the sum of the contribution of each fundamental frequency.

If kinematics are re-written as

$$\mathbf{u}(t) = U(\omega)\, e^{i\omega t} \quad (2)$$
$$\dot{\mathbf{u}}(t) = i\omega\, U(\omega)\, e^{i\omega t} \quad (3)$$
$$\ddot{\mathbf{u}}(t) = -\omega^2 U(\omega)\, e^{i\omega t} \quad (4)$$

$$\mathbf{F}(x,y,z,t) = \mathbf{F}(\omega)\, e^{i\omega t} \quad (5)$$

The dynamical equations of motion become:

$$\left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right]\cdot \mathbf{U}(\omega) = \mathbf{F}(\omega) \quad (6)$$

The system of equations (6) can be solved for each frequency, ω, and the corresponding amplitudes, $U(\omega)$, thus be obtained.

Another interesting application of the frequency domain approach (of substantial practical interest) is the estimation of natural frequencies from a known acceleration time history, often obtained experimentally. This can be achieved by using the properties of the discrete Fourier transform (7) and inverse Fourier transform (8) (Figure 1).

$$H_n = \sum_{k=0}^{N-1} h_k e^{-2\pi ikn/N} \quad (7)$$

$$h_k = \frac{1}{N}\sum_{n=0}^{N-1} H_n e^{2\pi ikn/N} \quad (8)$$

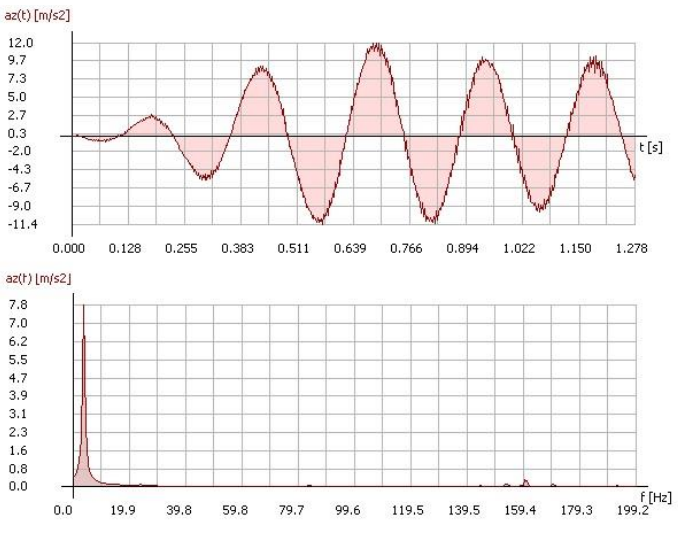

Figure 1. Frequency domain & Time domain

2.2 *Time Domain Analysis*

The time domain response can be evaluated employing three numerical techniques:

- Modal superposition analysis
- Implicit integration
- Explicit integration

2.2.1 Modal superposition analysis

In the modal superposition analysis, the system of differential equations of motion (1) is converted into a set of independent differential equations using the orthogonality properties of the mass and stiffness matrices in relation to the vibration modes. This decomposition if achieved by solving the following eigenvalue and eigenvector problem:

$$\det\left[\mathbf{K} - w^2\ \mathbf{M}\right] = \mathbf{0} \quad (9)$$

The solution of (9) will provide the structure's natural frequencies(eigenvalues), ω_n, and the corresponding vibration modes, ϕ_n (eigenvectors). The following generalised variables can be determined from the knowledge of the vibration modes, ϕ_n, as shown below

$$M_n = \boldsymbol{\phi}_n^T \cdot \mathbf{M} \cdot \boldsymbol{\phi}_n \quad (10)$$
$$C_n = \boldsymbol{\phi}_n^T \cdot \mathbf{C} \cdot \boldsymbol{\phi}_n \quad (11)$$
$$k_n = \boldsymbol{\phi}_n^T \cdot \mathbf{M} \cdot \boldsymbol{\phi}_n \quad (12)$$
$$F_n(x,y,z,t) = \boldsymbol{\phi}_n^T \cdot \mathbf{F}(x,y,z,t) \quad (13)$$

The response of each modal coordinate, $Y_n(t)$, is obtained from the solution of the corresponding decoupled equation

$$M_n \cdot \ddot{Y}_n(t) + C_n \cdot \dot{Y}_n(t) + k_n \cdot Y_n(t) = F_n(x,y,z,t) \quad (14)$$

This is usually achieved using a numerical integration method such as the Duhamel integral.

The response of the structure is finally obtained by superimposing the response of each modal coordinate, $Y_n(t)$, as follows:

$$\mathbf{u}(t) = \sum_{n=1}^{N\ Modes} \boldsymbol{\phi}_n \times Y_n(t) \quad (15)$$

The main advantage of the modal analysis is the fact that, although in theory we would need to consider all the vibration modes of the structure ($N_{Modes} = N_{DOF}$), for most bridges the contribution of only a small number of frequencies (modes) is usually enough to obtain an accurate response.

Although the determination of the natural frequencies and vibration modes (Eigenvalues and Eigen vectors) of a bridge structure usually involves substantial computational effort, once completed, allows for very a quick computation of the time domain response, as long as only reasonably small number of vibration modes are required.

2.2.2 Implicit and Explicit time integration

When integrating the equations of motion, we can distinguish two main approaches, the explicit and the implicit integration schemes.

The explicit methods use the differential equation at time "t" to predict the solution at time "t+Δt". This way the response of the system at "t+Δt" is "explicitly" obtained from the response at "t".

This procedure has the big advantage of not involving the solution of a set of linear equations for each time step. The weakness of this strategy lays on the fact the size of the time step, Δt, that guarantees the stability of the solution has to be very small. Nevertheless, when solving real problems that involve a vast number of degrees of freedom this technique is often the only one that can be employed.

The implicit methods attempt to satisfy the differential equation at time "t" from the solution of "t-Δt". This method implies the solution of a set of linear equations for each time step which can, in some cases, be a major disadvantage. However, solvers that take advantage of the 21st Century multi-core processor technology tend to improve computational efficiency considerably. Finally, it should be emphasized here that considerably larger time steps might be used, being some implicit methods unconditionally stable.

Newmark (Newmark 1969) has introduced a family of integration methods based on the following equations:

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t\, \dot{\mathbf{u}}_{t-\Delta t} + \left(\frac{1}{2} - \beta\right)\Delta t^2\, \ddot{\mathbf{u}}_{t-\Delta t} + \beta\,\Delta t^2\, \ddot{\mathbf{u}}_t \quad (16)$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + (1-\gamma)\,\Delta t\, \ddot{\mathbf{u}}_{t-\Delta t} + \gamma\,\Delta t\, \ddot{\mathbf{u}}_t \quad (17)$$

Depending on the γ and β parameters adopted we can have the following sub-families of implicit and explicit methods.

Table 1. Time integration methods.

γ	β	Time integration methods		
1/2	1/4	Average acceleration	Unconditionally stable	Implicit
1/2	1/6	Linear acceleration	Conditionally stable	Implicit
1/2	1/12	Fox-Goodwin	Conditionally stable	Implicit
1/2	0	Central difference	conditionally stable	Explicit

For the average acceleration method, which will be used in the solution of case study presented, we have:

$$\mathbf{K}_{eff}\,\mathbf{u}_t = \mathbf{F}_t^{eff} \quad (18)$$

with,

$$\begin{aligned}\mathbf{F}_t^{eff} = \mathbf{F}_t + \mathbf{M}\left(a_0\,\mathbf{u}_{t-\Delta t} + a_2\,\dot{\mathbf{u}}_{t-\Delta t} + a_3\,\ddot{\mathbf{u}}_{t-\Delta t}\right) + \\ \mathbf{C}\left(a_1\,\mathbf{u}_{t-\Delta t} + a_4\,\dot{\mathbf{u}}_{t-\Delta t} + a_5\,\ddot{\mathbf{u}}_{t-\Delta t}\right)\end{aligned} \quad (19)$$

$$\mathbf{K}_{eff}\,\mathbf{u}_t = \mathbf{F}_t^{eff} \quad (20)$$

And the coefficients a_0 to a_7 defined below.

Table 2. Newmark implicit integration parameters.

$a_0 = \frac{4}{\Delta t^2}$	$a_1 = \frac{2}{\Delta t}$	$a_2 = \frac{4}{\Delta t}$
$a_3 = 1$	$a_4 = 1$	$a_5 = 0$
$a_6 = \frac{\Delta t}{2}$	$a_7 = \frac{\Delta t}{2}$	

2.3 *VIFEM software*

VIFEM (VIFEM 2016) is Finite Element based software, coded in Java language, that results from the author's 18 years of research and development in the fields of structural dynamics and finite element modelling.

This software encompasses implementations of a number of the numerical analysis techniques covered in the preceding paragraphs. These include dynamic memory allocation, parallelisation of the solvers used both in the implicit and explicit integration of the dynamic equations of motion and in the determination of eigenvalues and eigenvectors.

VIFEM also has a graphical interface to allow the visualization of the all the results in a simple an interactive way, via the extensive use of graphs and animations.

3 CASE STUDY

3.1 *Introduction*

In order to illustrate the relative performance of the different numerical analysis techniques described before when modelling the dynamic response of a bridge to moving vehicular loads, a real bridge was therefore computed. The case study chosen was a 52.0m span, 16.7m wide simply supported bridge with steel-concrete composite deck, located in Somerset, UK. This bridge comprises 6 No. steel girders at 2800mm spacing, topped by a 225 mm slab a (Figure 2).

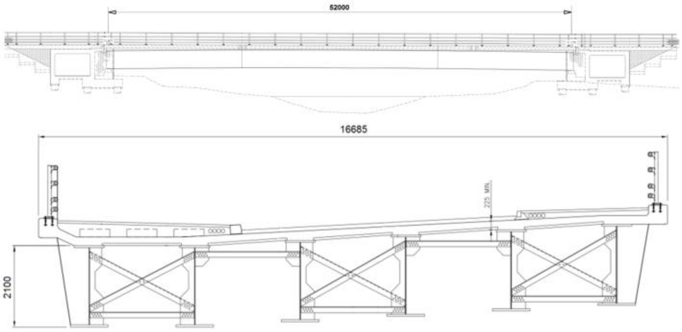

Figure 2. Somerset Bridge. Typical cross section.

3.2 *Loading*

The vehicular load adopted in the numerical analysis was a five axle 44 ton vehicle crossing the bridge at a speed of 60 Km/h (Figure 3).

Figure 3. Somerset Bridge. 5 Axle 44 ton vehicle layout.

3.3 *Finite Element model*

The bridge was modelled with a 3D finite element mesh made out of a combination of 2996 Euler-Bernoulli beam elements (girder flanges and bracing) and 8832 4 node thick shell elements (reinforced concrete deck and girder webs), totalizing 60336 degrees of freedom (DOF) (Figure 4).

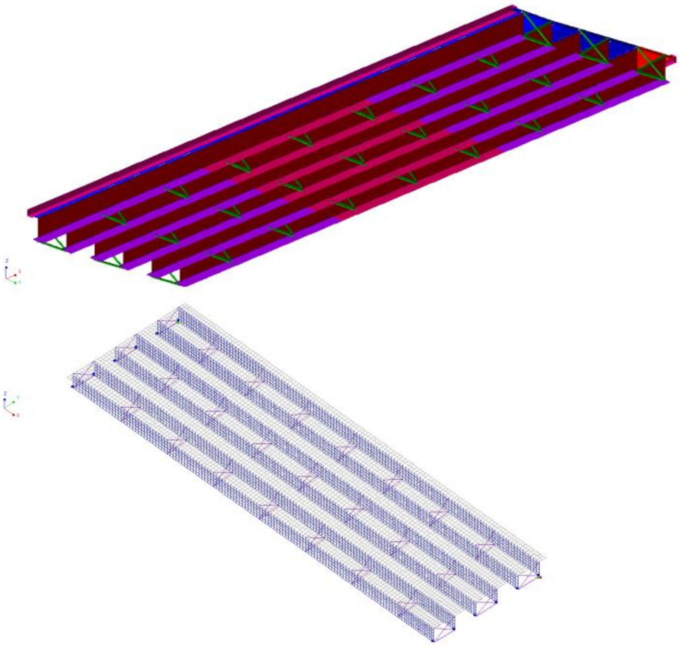

Figure 4. Somerset Bridge. 3D Finite Element mesh.

3.4 *Dynamic response analysis*

The dynamic response was carried out using both implicit Newmark integration and modal superposition using software VIFEM. The time step adopted, Δt, was 0.02 s and 256 time steps were considered. A damping coefficient of 5% was taken for the concrete and a 2% damping coefficient was taken for the steel members. Proportional Rayleigh damping was considered based on two control frequencies, f_1 and f_2, equal to 1 Hz and 15 Hz, respectively.

3.4.1 Time integration analysis

The typical output of analysis obtained with VIFEM is shown in Figure 5, where the response of a node in the “z” direction is plotted both in terms of displacements, $u_z(t)$ (in blue), velocities $v_z(t)$ (in green) and accelerations, $a_z(t)$ (in red).

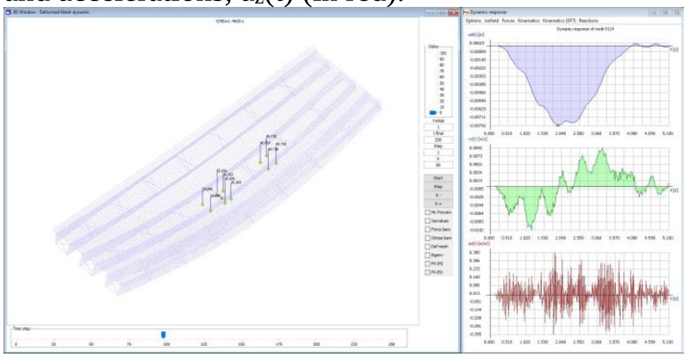

Figure 5. VIFEM - Typical output.

Alternatively, it is also possible to plot the frequency domain response $u_z(f)$, $v_z(f)$, $a_z(f)$ (via the use of a discrete Fourier transform) side-by-side with the time domain response. (Figure 6). By doing so, becomes apparent the relative importance of each frequency to the overall response of the “node” in study.

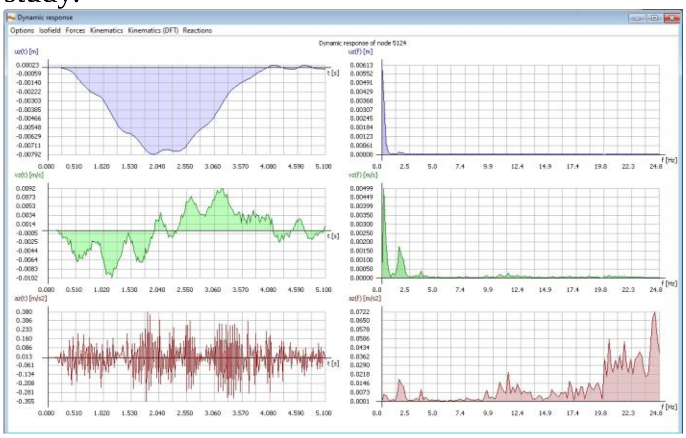

Figure 6. VIFEM - Time domain & Frequency domain output.

This can be particularly useful to estimate the natural frequencies associated with that degree of freedom, without the requirement for an eigenvalue and eigenvector analysis to be performed.

3.4.2 Modal superposition analysis

The modal superposition analysis was carried out by firstly computing the 13 first natural frequencies, ω_n, (Table 3) and corresponding modal shapes, $\phi_{n.}$ of the bridge (Figure 7) and, subsequently, superimposing the response of each mode, using equation (15).

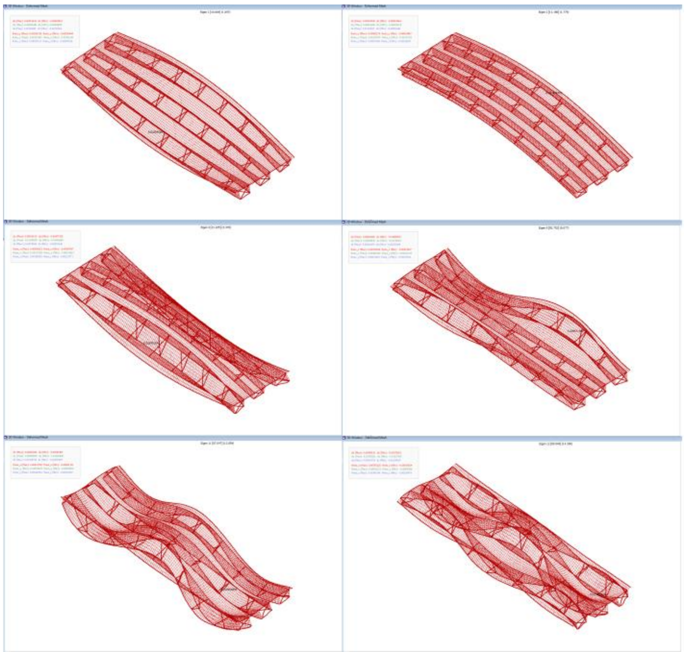

Figure 7. Modal analysis – 6 Typical modal shapes

Table 3. Natural frequencies.

Mode	w_n [rad/s]	f_n [Hz]
1	10.66	1.70
2	11.18	1.78
3	21.22	3.38
4	27.77	4.42
5	39.76	6.33
6	44.02	7.01
7	45.35	7.22
8	50.75	8.08
9	58.19	9.26
10	84.13	13.39
11	87.05	13.85
12	89.85	14.30
13	93.04	14.81

3.4.3 *Analysis of results*

The following graphs show a comparison between the dynamic response obtained via implicit integration and modal superposition for a node located close to the mid span. Additionally, the pseudo-static displacement resulting from positioning the 44 ton 5 axle vehicle at the location corresponding to the maximum dynamic displacement is also shown.

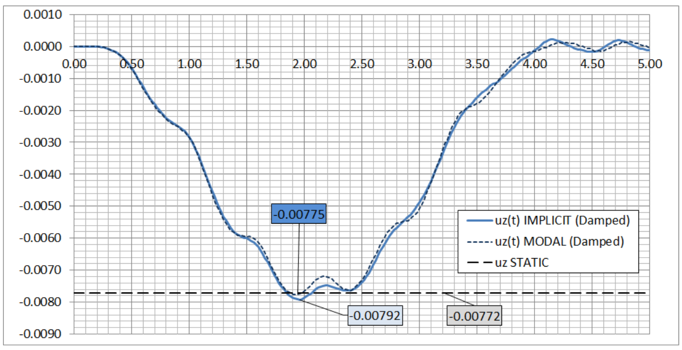

Figure 8. – Implicit integration & Modal analysis (u_z)

The dynamic amplification factor, DAF, obtained from comparing the results of the implicit integration with the pseudo-static analysis is circa 2.6 %. Using the results of the Modal analysis, the DAF is 1.0%.

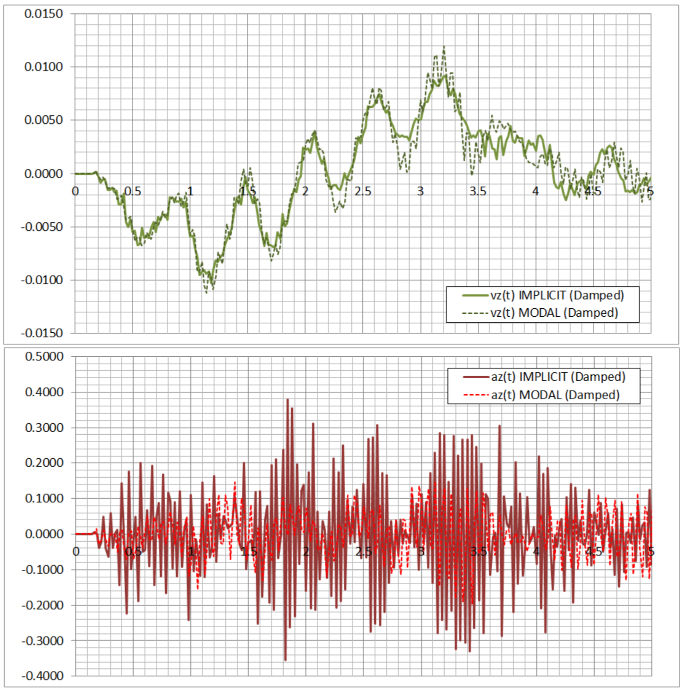

Figure 9. – Implicit integration & Modal analysis (v_z and a_z)

4 CONCLUSIONS

The analysis carried out on Somerset Bridge shows that it is possible to run fully implicit dynamic analysis of life size bridges to vehicular moving loading, even when the number of degrees of freedom (DOF) amounts to tens of thousands. With the use of multi-core processing oriented routines, the analysis can be completed within a reasonable time frame.

Modal superposition analysis can be an attractive alternative, particularly when a relatively small number of frequencies are required to obtain an accurate response.

The dynamic amplification (DAF) of the displacements (and derived internal stresses) resulting from the moving vehicular load was less than 3%. This seems to indicate that even the most modern design codes are over conservative.

5 REFERENCES

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